Jacobian conjecture and nilpotency

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Abstract

For K a field of characteristic 0 and d any integer number ≥ 2, we prove the invertibility of polynomial maps $F : K^d \to K^d$ of the form $F = Id + H$, where each H_i is the cube of a linear form and the Jacobian matrix JH satisfies $(JH)^3 = 0$. Our proof uses the inversion algorithm for polynomial maps presented in our previous paper. Our current result leads us to formulate a conjecture relating the nilpotency degree of the matrix JH with the number of necessary steps in the inversion algorithm.

Keywords: Polynomial automorphism, Jacobian Problem, inversion algorithm.
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1 Introduction

The Jacobian Conjecture originated in the question raised by Keller in [6] on the invertibility of polynomial maps with Jacobian determinant equal to 1. The question is still open in spite of the efforts of many mathematicians. We recall in the sequel the precise statement of the Jacobian Conjecture, some reduction theorems and other results we shall use. We refer to [5] for a detailed account of the research on the Jacobian Conjecture and related topics.

Let K be a field and $K[X] = K[X_1, \dots, X_d]$ the polynomial ring in the variables $X_1, \dots, X_d$ over K. A *polynomial map* is a map $F = (F_1, \dots, F_d) : K^d \to K^d$ of the form

$$(X_1, \dots, X_d) \mapsto (F_1(X_1, \dots, X_d), \dots, F_d(X_1, \dots, X_d)),$$

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where $F_i \in K[X], 1 \leq i \leq d$. The polynomial map F is *invertible* if there exists a polynomial map $G = (G_1, \ldots, G_d) : K^d \to K^d$ such that $X_i = G_i(F_1, \ldots, F_d), 1 \leq i \leq d$. We shall call F a *Keller map* if the Jacobian matrix

$$JF = \left(\frac{\partial F_i}{\partial X_j}\right)_{\substack{1 \leq i \leq d \\ 1 \leq j \leq d}}$$

has determinant equal to 1. Clearly an invertible polynomial map F has a Jacobian matrix JF with non zero determinant and may be transformed into a Keller map by composition with the linear automorphism with matrix $JF(0)^{-1}$.

Jacobian Conjecture. *Let K be a field of characteristic zero. A Keller map $F : K^d \to K^d$ is invertible.*

In the sequel, K will always denote a **field of characteristic zero.**

For $F = (F_1, \ldots, F_d) \in K[X]^d$, we define the *degree* of F as $\deg F = \max\{\deg F_i : 1 \leq i \leq d\}$. It is known that if F is a polynomial automorphism, then $\deg F^{-1} \leq (\deg F)^{d-1}$ (see [2] or [7]).

The Jacobian conjecture for quadratic maps was proved by Wang in [8]. We state now the reduction of the Jacobian conjecture to the case of maps of third degree (see [2], [9], [3] and [4]).

Proposition 1. *a) (Bass-Connell-Wright-Yagzhev) Given a Keller map $F : \mathbb{C}^d \to \mathbb{C}^d$, there exists a Keller map $\widetilde{F} : \mathbb{C}^D \to \mathbb{C}^D$, $D \geq d$ of the form $\widetilde{F} = Id + H$, where $H(X)$ is a homogeneous cubic map and having the following property: if $\widetilde{F}$ is invertible, then F is invertible too.*

b) (Drużkowski) The cubic part H may be chosen of the form

$$\left((\sum_{j=1}^{D} a_{1j} X_j)^3, \ldots, (\sum_{j=1}^{D} a_{Dj} X_j)^3 \right)$$

and with the matrix $A = (a_{ij})_{\substack{1 \leq i \leq D \\ 1 \leq j \leq D}}$ satisfying $A^2 = 0$.

Let us note that for a polynomial map in d variables of the form $F = Id + H$, with H homogeneous of degree at least two, the condition $det(JF) = 1$ is equivalent to $(JH)^d = 0$. Given a polynomial map $F : \mathbb{C}^d \to \mathbb{C}^d$, we shall call a polynomial $P \in \mathbb{C}[X_1, \ldots, X_d]$ *invariant under* F if $P(F_1, \ldots, F_d) = P(X_1 \ldots, X_d)$. A polynomial map $F : \mathbb{C}^d \to \mathbb{C}^d$ of the form $F = Id + H$ is called a *quasi-translation* if $F^{-1} = Id - H$.

In [1] we proved the following algorithmic equivalent statement to the Jacobian conjecture for homogeneous polynomial maps. Although it is there stated in the complex case, it is clearly valid for any field K of characteristic 0.

Theorem 2. *Let $F : K^d \to K^d$ be a polynomial map of the form*

$$\begin{cases} F_1(X_1,\ldots,X_d) &=& X_1 + H_1(X_1,\ldots,X_d) \\ F_2(X_1,\ldots,X_d) &=& X_2 + H_2(X_1,\ldots,X_d) \\ & \vdots & \\ F_d(X_1,\ldots,X_d) &=& X_d + H_d(X_1,\ldots,X_d), \end{cases}$$

where $H_i(X_1,\ldots,X_d)$ is a homogeneous polynomial in $X_1,\ldots,X_d$, of degree 3, $1 \le i \le d$. For each $i = 1,\ldots,d$, we consider the polynomial sequence (P_j^i) defined in the following way

$$\begin{array}{rcl} P_0^i(X_1,\ldots,X_d) &=& X_i, \\ P_1^i(X_1,\ldots,X_d) &=& H_i, \end{array}$$

and, assuming P_{j-1}^i is defined,

$$P_j^i(X_1,\ldots,X_d) = P_{j-1}^i(F_1,\ldots,F_d) - P_{j-1}^i(X_1,\ldots,X_d).$$

Then F is invertible if and only if for all $i = 1,\ldots,d$, there exists an integer m_i such that $P_{m_i}^i = 0$. In this case, the inverse map G of F is given by

$$G_i(Y_1,Y_2,\ldots,Y_d) = \sum_{l=0}^{m_i-1} (-1)^l P_l^i(Y_1,Y_2,\ldots,Y_d),\ 1 \le i \le d.$$

In this paper we shall use this theorem to prove that a polynomial map of the Drużkowski form such that the Jacobian matrix of H is nilpotent of degree at most 3 is invertible.

2 Main result

We shall prove the following theorem

Theorem 3. *Let $F : K^d \to K^d$ be a polynomial map of the form*

$$\begin{cases} F_1(X_1,\ldots,X_d) &=& X_1 + H_1(X_1,\ldots,X_d) \\ F_2(X_1,\ldots,X_d) &=& X_2 + H_2(X_1,\ldots,X_d) \\ & \vdots & \\ F_d(X_1,\ldots,X_d) &=& X_d + H_d(X_1,\ldots,X_d), \end{cases}$$

where $H_i(X_1,\ldots,X_d) = L_i(X_1,\ldots,X_d)^3$ and $L_i(X_1,\ldots,X_d) = a_{i1}X_1 + \cdots + a_{id}X_d$, $1 \le i \le d$. We consider the jacobian matrix JH of $H := (H_1,\ldots,H_d)$

$$JH = \begin{pmatrix} \dfrac{\partial H_1}{\partial X_1} & \cdots & \dfrac{\partial H_1}{\partial X_d} \\ & \vdots & \\ \dfrac{\partial H_d}{\partial X_1} & \cdots & \dfrac{\partial H_d}{\partial X_d} \end{pmatrix}.$$

If $(JH)^3=0$*, then* F *is invertible and* $\deg F^{-1}\leq 9$*.*

The theorem follows from Theorem 2 and the Proposition 4 below.

Proposition 4. *Let* F *be as in Theorem 3 and let the polynomials* P_j^i *be defined as in Theorem 2. If* $(JH)^3=0$*, then* $\forall i=1,\dots,d$*,* $\deg P_j^i\leq 9$*, for* $j=2,3,4$*, and* $P_5^i=0$*.*

Proof of Theorem 3 assuming Proposition 4. If $(JH)^3=0$, we have $P_5^i=0$, for all i, by Proposition 4 and this implies, by Theorem 2, that F is invertible and the inverse map G of F is given by $G_i(Y_1,Y_2,\dots,Y_d)=Y_i-P_1^i(Y_1,Y_2,\dots,Y_d)+P_2^i(Y_1,Y_2,\dots,Y_d)-P_3^i(Y_1,Y_2,\dots,Y_d)+P_4^i(Y_1,Y_2,\dots,Y_d)$. By the bound on the degrees given by Proposition 4, we obtain $\deg F^{-1}\leq 9$.

Proof of Proposition 4. This proof is purely computational. We shall use the expression of the polynomials P_j^i as a sum of homogeneous polynomials obtained in [1], proof of Theorem 4. From the condition $(JH)^3=0$, we shall derive some equations which will allow to prove the vanishing of some homogeneous summands of the polynomials P_j^i.

Since $(JH)^3=0$, we have

$$\sum_{j,k=1}^{d}\frac{\partial H_i}{\partial X_j}\frac{\partial H_j}{\partial X_k}\frac{\partial H_k}{\partial X_l}=0,\quad \forall\, i,l=1,\dots,d.$$

i.e.

$$\sum_{j,k=1}^{d}(3L_i^2a_{ij})(3L_j^2a_{jk})(3L_k^2a_{kl})=27L_i^2(\sum_{j,k=1}^{d}a_{ij}L_j^2a_{jk}L_k^2a_{kl})=0,\quad \forall\, i,l=1,\dots,d.$$

which implies

$$\sum_{j,k=1}^{d}a_{ij}a_{jk}a_{kl}L_j^2L_k^2=0,\quad \forall\, i,l=1,\dots,d, \tag{1}$$

since $L_i=0$ implies $a_{ij}=0$, for all j, hence (1). Applying $\dfrac{\partial}{\partial X_m}$, we obtain

$$\sum_{j,k=1}^{d}a_{ij}a_{jk}a_{kl}(a_{jm}L_jL_k^2+a_{km}L_j^2L_k)=0,\quad \forall\, i,l,m=1,\dots,d. \tag{2}$$

Further, from (1), we obtain

$$\sum_{j,k=1}^{d}a_{ij}a_{jk}L_j^2L_k^3=0,\quad \forall\, i=1,\dots,d \tag{3}$$

since $\sum_{j,k=1}^{d}a_{ij}a_{jk}L_j^2L_k^3=\sum_{j,k=1}^{d}a_{ij}a_{jk}L_j^2L_k^2(\sum_{l=1}^{d}a_{kl}X_l)=\sum_{l=1}^{d}(\sum_{j,k=1}^{d}a_{ij}a_{jk}a_{kl}L_j^2L_k^2)X_l$.
Now, from (2), we obtain similarly

$$\sum_{j,k=1}^{d} a_{ij}a_{jk}(a_{jm}L_jL_k^3 + a_{km}L_j^2L_k^2) = 0, \quad \forall\, i, m = 1, \ldots, d$$

and, using (1)

$$\sum_{j,k=1}^{d} a_{ij}a_{jk}a_{jm}L_jL_k^3 = 0, \quad \forall\, i, m = 1, \ldots, d \tag{4}$$

Applying $\partial/\partial X_n$ to (4), we obtain

$$\sum_{j,k=1}^{d} a_{ij}a_{jk}a_{jm}(a_{jn}L_k^3 + 3a_{kn}L_jL_k^2) = 0$$

i.e.

$$\sum_{j,k=1}^{d} a_{ij}a_{jk}a_{jm}a_{jn}L_k^3 = -3\sum_{j,k=1}^{d} a_{ij}a_{jk}a_{jm}a_{kn}L_jL_k^2 \tag{5}$$

We shall use equations (3), (4) and (5) repeatedly to prove the vanishing of some homogeneous summands of the polynomials P_2^i, P_3^i, P_4^i and the vanishing of P_5^i. Since the calculations are very similar we will detail them only the first time we apply each of these equations.
We have

$$P_2^i = H_i(F) - H_i(X) = H_i(X+H) - H_i(X) = Q_{21}^i + Q_{22}^i + Q_{23}^i$$

where

$$\begin{aligned}
Q_{21}^i &= \textstyle\sum_{j=1}^d H_j \dfrac{\partial H_i}{\partial X_j} = 3L_i^2 \sum_{j=1}^d a_{ij}L_j^3 \\
Q_{22}^i &= \dfrac{1}{2}\textstyle\sum_{1\le j,k\le d} H_jH_k \dfrac{\partial^2 H_i}{\partial X_j \partial X_k} = 3L_i \sum_{1\le j,k\le d} a_{ij}a_{ik}L_j^3L_k^3 \\
Q_{23}^i &= \dfrac{1}{6}\textstyle\sum_{1\le j,k,l\le d} H_jH_kH_l \dfrac{\partial^3 H_i}{\partial X_j \partial X_k \partial X_l} = \sum_{1\le j,k,l\le d} a_{ij}a_{ik}a_{il}L_j^3L_k^3L_l^3,
\end{aligned}$$

hence, in particular $\deg P_2^i \le 9$.

In order to determine P_3^i, we need to compute the derivatives of Q_{21}^i, Q_{22}^i and Q_{23}^i. We compute first the derivatives of Q_{21}^i.

$$\begin{aligned}
\frac{\partial Q_{21}^i}{\partial X_k} &= 6a_{ik}L_i \textstyle\sum_{j=1}^d a_{ij}L_j^3 + 9L_i^2 \sum_{j=1}^d a_{ij}a_{jk}L_j^2 \\
\frac{\partial^2 Q_{21}^i}{\partial X_k \partial X_l} &= 6a_{ik}a_{il} \textstyle\sum_{j=1}^d a_{ij}L_j^3 + 18L_ia_{ik}\sum_{j=1}^d a_{ij}a_{jl}L_j^2 \\
&\quad +18L_ia_{il}\textstyle\sum_{j=1}^d a_{ij}a_{jk}L_j^2 + 18L_i^2\sum_{j=1}^d a_{ij}a_{jk}a_{jl}L_j
\end{aligned}$$

The derivative $\partial^3 Q_{21}^i/\partial X_k \partial X_l \partial X_m$ is a sum of terms of one of the following forms

$$18 a_{ik} a_{il} \sum_{j=1}^{d} a_{ij} a_{jm} L_j^2, \quad 36 L_i a_{ik} \sum_{j=1}^{d} a_{ij} a_{jl} a_{jm} L_j,$$

up to a permutation of the set $\{k, l, m\}$, and the summand $18 L_i^2 \sum_{j=1}^{d} a_{ij} a_{jk} a_{jl} a_{jm}$. The derivative $\partial^4 Q_{21}^i/\partial X_k \partial X_l \partial X_m \partial X_n$ is a sum of terms of one of the following forms

$$36 a_{ik} a_{il} \sum_{j=1}^{d} a_{ij} a_{jm} a_{jn} L_j, \quad 36 L_i a_{ik} \sum_{j=1}^{d} a_{ij} a_{jl} a_{jm} a_{jn},$$

up to a permutation of the set $\{k, l, m, n\}$. The derivative $\partial^5 Q_{21}^i/\partial X_k \partial X_l \partial X_m \partial X_n \partial X_p$ is a sum of terms of the form

$$36 a_{ik} a_{il} \sum_{j=1}^{d} a_{ij} a_{jm} a_{jn} a_{jp}$$

up to a permutation of the set $\{k, l, m, n, p\}$.

We compute now the derivatives of Q_{22}^i.

$$\begin{aligned} \frac{\partial Q_{22}^i}{\partial X_l} &= 3 a_{il} \textstyle\sum_{1 \le j,k \le d} a_{ij} a_{ik} L_j^3 L_k^3 + 9 L_i \sum_{1 \le j,k \le d} a_{ij} a_{ik} a_{jl} L_j^2 L_k^3 \\ &\quad + 9 L_i \textstyle\sum_{1 \le j,k \le d} a_{ij} a_{ik} a_{kl} L_j^3 L_k^2 \end{aligned}$$

The derivative $\partial^2 Q_{22}^i/\partial X_l \partial X_m$ is a sum of terms of one of the following forms

$$\begin{array}{ll} 9 a_{il} \sum_{1 \le j,k \le d} a_{ij} a_{ik} a_{jm} L_j^2 L_k^3, & 18 L_i \sum_{1 \le j,k \le d} a_{ij} a_{ik} a_{jl} a_{jm} L_j L_k^3 \\ 27 L_i \sum_{1 \le j,k \le d} a_{ij} a_{ik} a_{jl} a_{km} L_j^2 L_k^2, & \end{array}$$

up to switching l with m, and j with k. The derivative $\partial^3 Q_{22}^i/\partial X_l \partial X_m \partial X_n$ is a sum of terms of one of the following forms

$$\begin{array}{ll} 18 a_{il} \sum_{1 \le j,k \le d} a_{ij} a_{ik} a_{jm} a_{jn} L_j L_k^3, & 27 a_{il} \sum_{1 \le j,k \le d} a_{ij} a_{ik} a_{jm} a_{kn} L_j^2 L_k^2, \\ 18 L_i \sum_{1 \le j,k \le d} a_{jn} a_{ij} a_{ik} a_{jl} a_{jm} L_k^3, & 54 L_i \sum_{1 \le j,k \le d} a_{ij} a_{ik} a_{jl} a_{jm} a_{kn} L_j L_k^2, \end{array}$$

up to a permutation of the set $\{l, m, n\}$ and a switch of j with k. The derivative $\partial^4 Q_{22}^i/\partial X_l \partial X_m \partial X_n \partial X_p$ is a sum of terms of one of the following forms

$$\begin{array}{ll} 18 a_{il} \sum_{1 \le j,k \le d} a_{ij} a_{ik} a_{jm} a_{jn} a_{jp} L_k^3, & 54 a_{il} \sum_{1 \le j,k \le d} a_{ij} a_{ik} a_{jm} a_{jn} a_{kp} L_j L_k^2, \\ 54 L_i \sum_{1 \le j,k \le d} a_{ij} a_{ik} a_{jl} a_{jm} a_{jn} a_{kp} L_k^2, & 108 L_i \sum_{1 \le j,k \le d} a_{ij} a_{ik} a_{jl} a_{jm} a_{kn} a_{kp} L_j L_k, \end{array}$$

up to a permutation of the set $\{l, m, n, p\}$ and a switch of j with k. The derivative $\partial^5 Q_{22}^i/\partial X_l \partial X_m \partial X_n \partial X_p \partial X_q$ is a sum of terms of one of the following forms

$$\begin{array}{ll} 54 a_{il} \sum_{1 \le j,k \le d} a_{ij} a_{ik} a_{jm} a_{jn} a_{jp} a_{kq} L_k^2, & 108 a_{il} \sum_{1 \le j,k \le d} a_{ij} a_{ik} a_{jm} a_{jn} a_{kp} a_{kq} L_j L_k, \\ 108 L_i \sum_{1 \le j,k \le d} a_{ij} a_{ik} a_{jl} a_{jm} a_{jn} a_{kp} a_{kq} L_k, & \end{array}$$

up to a permutation of the set $\{l,m,n,p,q\}$ and a switch of j with k. The derivative $\partial^6 Q^i_{22}/\partial X_l\partial X_m\partial X_n\partial X_p\partial X_q\partial X_r$ is a sum of terms of one of the following forms

$$108a_{il}\sum_{1\le j,k\le d} a_{ij}a_{ik}a_{jm}a_{jn}a_{jp}a_{kq}a_{kr}L_k,\quad 108L_i\sum_{1\le j,k\le d} a_{ij}a_{ik}a_{jl}a_{jm}a_{jn}a_{kp}a_{kq}a_{kr},$$

up to a permutation of the set $\{l,m,n,p,q,r\}$ and a switch of j with k. The derivative $\partial^7 Q^i_{22}/\partial X_l\partial X_m\partial X_n\partial X_p\partial X_q\partial X_r\partial X_s$ is a sum of terms of the form

$$108a_{il}\sum_{1\le j,k\le d} a_{ij}a_{ik}a_{jm}a_{jn}a_{jp}a_{kq}a_{kr}a_{ks},$$

up to a permutation of the set $\{l,m,n,p,q,r,s\}$ and a switch of j with k.

Finally we compute the derivatives of Q^i_{23}.

$$\frac{\partial Q^i_{23}}{\partial X_m} = 3\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}L_j^2L_k^3L_l^3 + 3\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{km}L_j^3L_k^2L_l^3 + 3\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{lm}L_j^3L_k^3L_l^2.$$

The derivative $\partial^2 Q^i_{23}/\partial X_m\partial X_n$ is a sum of terms of one of the following forms

$$6\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}L_jL_k^3L_l^3,\quad 9\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{kn}L_j^2L_k^2L_l^3$$

up to a permutation of the set $\{m,n\}$ and a permutation of the set $\{i,j,k\}$. The derivative $\partial^3 Q^i_{23}/\partial X_m\partial X_n\partial X_p$ is a sum of terms of one of the following forms

$$\begin{array}{ll} 6\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}L_k^3L_l^3, & 18\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{kp}L_jL_k^2L_l^3 \\ 27\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{kn}a_{lp}L_j^2L_k^2L_l^2, & \end{array}$$

up to a permutation of the set $\{m,n,p\}$ and a permutation of the set $\{i,j,k\}$. The derivative $\partial^4 Q^i_{23}/\partial X_m\partial X_n\partial X_p\partial X_q$ is a sum of terms of one of the following forms

$$\begin{array}{ll} 18\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}L_k^2L_l^3, & 36\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{kp}a_{kq}L_jL_kL_l^3 \\ 54\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{kp}a_{lq}L_jL_k^2L_l^2, & \end{array}$$

up to a permutation of the set $\{m,n,p,q\}$ and a permutation of the set $\{i,j,k\}$. The derivative $\partial^5 Q^i_{23}/\partial X_m\partial X_n\partial X_p\partial X_q\partial X_r$ is a sum of terms of one of the following forms

$$\begin{array}{ll} 36\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}a_{kr}L_kL_l^3, & 54\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}a_{lr}L_k^2L_l^2 \\ 108\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{kp}a_{kq}a_{lr}L_jL_kL_l^2, & \end{array}$$

up to a permutation of the set $\{m,n,p,q,r\}$ and a permutation of the set $\{i,j,k\}$. The derivative $\partial^6 Q^i_{23}/\partial X_m\partial X_n\partial X_p\partial X_q\partial X_r\partial X_s$ is a sum of terms of one of the following forms

$36\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}a_{kr}a_{ks}L_l^3$, $\quad 108\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}a_{lr}a_{ks}L_kL_l^2$
$216\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{kp}a_{kq}a_{lr}a_{ls}L_jL_kL_l$,

up to a permutation of the set $\{m,n,p,q,r,s\}$ and a permutation of the set $\{i,j,k\}$. The derivative $\partial^7 Q_{23}^i/\partial X_m\partial X_n\partial X_p\partial X_q\partial X_r\partial X_s\partial X_t$ is a sum of terms of one of the following forms

$$108\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}a_{kr}a_{ks}a_{lt}L_l^2, \quad 216\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}a_{lr}a_{ks}a_{lt}L_kL_l$$

up to a permutation of the set $\{m,n,p,q,r,s,t\}$ and a permutation of the set $\{i,j,k\}$. The derivative $\partial^8 Q_{23}^i/\partial X_m\partial X_n\partial X_p\partial X_q\partial X_r\partial X_s\partial X_t\partial X_u$ is a sum of terms of the following form

$$216\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}a_{kr}a_{ks}a_{lt}a_{lu}L_l$$

up to a permutation of the set $\{m,n,p,q,r,s,t,u\}$ and a permutation of the set $\{i,j,k\}$. The derivative $\partial^9 Q_{23}^i/\partial X_m\partial X_n\partial X_p\partial X_q\partial X_r\partial X_s\partial X_t\partial X_u\partial X_v$ is a sum of terms of the following form

$$216\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}a_{kr}a_{ks}a_{lt}a_{lu}a_{lv}$$

up to a permutation of the set $\{m,n,p,q,r,s,t,u,v\}$ and a permutation of the set $\{i,j,k\}$.

We write $P_3^i = \sum_{k=1}^{11} Q_{3k}^i$, with Q_{3k}^i homogeneous of degree $5+2k$ (see [1] proof of Theorem 4). We have, using (3),

$$\begin{aligned} Q_{31}^i = \textstyle\sum_{k=1}^d \frac{\partial Q_{21}^i}{\partial X_k} H_k &= 6L_i \textstyle\sum_{j,k=1}^d a_{ik}a_{ij}L_j^3L_k^3 + 9L_i^2 \textstyle\sum_{j,k=1}^d a_{ij}a_{jk}L_j^2L_k^3 \\ &= 6L_i \textstyle\sum_{j,k=1}^d a_{ij}a_{ik}L_j^3L_k^3. \end{aligned}$$

Using (3) and (4), we obtain

$$Q_{32}^i = \frac{1}{2}\sum_{k,l} \frac{\partial^2 Q_{21}^i}{\partial X_k \partial X_l} H_k H_l + \sum_l \frac{\partial Q_{22}^i}{\partial X_l} H_l = 6\sum_{j,k,l} a_{ij}a_{ik}a_{il}L_j^3L_k^3L_l^3,$$

since

$$\begin{aligned} \textstyle\sum_{j,k,l} a_{ij}a_{ik}a_{jl}L_j^2L_k^3L_l^3 &= \textstyle\sum_k a_{ik}L_k^3(\sum_{j,l} a_{ij}a_{jl}L_j^2L_l^3) \overset{(3)}{=} 0 \\ \textstyle\sum_{j,k,l} a_{ij}a_{jk}a_{il}L_j^2L_k^3L_l^3 &= \textstyle\sum_l a_{il}L_l^3(\sum_{j,k} a_{ij}a_{jk}L_j^2L_k^3) \overset{(3)}{=} 0 \\ \textstyle\sum_{j,k,l} a_{ij}a_{jk}a_{jl}L_jL_k^3L_l^3 &= \textstyle\sum_l L_l^3(\sum_{j,k} a_{ij}a_{jk}a_{jl}L_jL_k^3) \overset{(4)}{=} 0 \\ \textstyle\sum_{j,k,l} a_{ij}a_{ik}a_{kl}L_j^3L_k^2L_l^3 &= \textstyle\sum_j a_{ij}L_j^3(\sum_{k,l} a_{ik}a_{kl}L_k^2L_l^3) \overset{(3)}{=} 0 \end{aligned}$$

Using again (3) and (4), we obtain

$$\begin{aligned} Q_{33}^i &= \frac{1}{6}\sum_{k,l,m} \frac{\partial^3 Q_{21}^i}{\partial X_k \partial X_l \partial X_m} H_k H_l H_m + \frac{1}{2}\sum_{l,m} \frac{\partial^2 Q_{22}^i}{\partial X_l \partial X_m} H_l H_m + \sum_m \frac{\partial Q_{23}^i}{\partial X_m} H_m \\ &= 0. \end{aligned}$$

We have now, using (3), (4) and (5),

$$Q_{34}^i = \frac{1}{4!}\sum_{k,l,m,n} \frac{\partial^4 Q_{21}^i}{\partial X_k \partial X_l \partial X_m \partial X_n} H_k H_l H_m H_n + \frac{1}{3!}\sum_{l,m,n} \frac{\partial^3 Q_{22}^i}{\partial X_l \partial X_m \partial X_n} H_l H_m H_n$$
$$+\frac{1}{2}\sum_{m,n} \frac{\partial^2 Q_{23}^i}{\partial X_m \partial X_n} H_m H_n = 0.$$

We detail the use of (5):

$$\begin{aligned} \sum_{j,k,l,m,n} a_{ij} a_{ik} a_{jl} a_{jm} a_{jn} L_k^3 L_l^3 L_m^3 L_n^3 &= \sum_{k,m,n} a_{ik} L_k^3 L_m^3 L_n^3 (\sum_{j,l} a_{ij} a_{jl} a_{jm} a_{jn} L_l^3) \\ &\overset{(5)}{=} -3\sum_{k,m,n} a_{ik} L_k^3 L_m^3 L_n^3 (\sum_{j,l} a_{ij} a_{jl} a_{jm} a_{ln} L_j L_l^2) \\ &= -3\sum_{k,l,n} a_{ik} a_{jl} L_k^3 L_l^2 L_n^3 (\sum_{j,m} a_{ij} a_{jl} a_{jm} L_j L_m^3) \overset{(4)}{=} 0 \end{aligned}$$

Similarly, using again (3), (4) and (5),

$$\begin{aligned} Q_{35}^i &= \frac{1}{5!}\sum_{k,l,m,n,p} \frac{\partial^5 Q_{21}^i}{\partial X_k \partial X_l \partial X_m \partial X_n \partial X_p} H_k H_l H_m H_n H_p \\ &\quad +\frac{1}{4!}\sum_{l,m,n,p} \frac{\partial^4 Q_{22}^i}{\partial X_l \partial X_m \partial X_n \partial X_p} H_l H_m H_n H_p \\ &\quad +\frac{1}{3!}\sum_{m,n,p} \frac{\partial^3 Q_{23}^i}{\partial X_m \partial X_n \partial X_p} H_m H_n H_p = 0 \end{aligned}$$

$$Q_{36}^i = \frac{1}{5!}\sum_{l,m,n,p,q} \frac{\partial^5 Q_{22}^i}{\partial X_l \partial X_m \partial X_n \partial X_p \partial X_q} H_l H_m H_n H_p H_q$$
$$+\frac{1}{4!}\sum_{m,n,p,q} \frac{\partial^4 Q_{23}^i}{\partial X_m \partial X_n \partial X_p \partial X_q} H_m H_n H_p H_q = 0$$

$$Q_{37}^i = \frac{1}{6!}\sum_{l,m,n,p,q,r} \frac{\partial^6 Q_{22}^i}{\partial X_l \partial X_m \partial X_n \partial X_p \partial X_q \partial X_r} H_l H_m H_n H_p H_q H_r$$
$$+\frac{1}{5!}\sum_{m,n,p,q,r} \frac{\partial^5 Q_{23}^i}{\partial X_m \partial X_n \partial X_p \partial X_q \partial X_r} H_m H_n H_p H_q H_r = 0$$

$$Q_{38}^i = \frac{1}{7!}\sum_{l,m,n,p,q,r,s} \frac{\partial^7 Q_{22}^i}{\partial X_l \partial X_m \partial X_n \partial X_p \partial X_q \partial X_r \partial X_s} H_l H_m H_n H_p H_q H_r H_s$$
$$+\frac{1}{6!}\sum_{m,n,p,q,r,s} \frac{\partial^6 Q_{23}^i}{\partial X_m \partial X_n \partial X_p \partial X_q \partial X_r \partial X_s} H_m H_n H_p H_q H_r H_s = 0$$

Using (3) and (4),

$$Q_{39}^i = \frac{1}{7!} \sum_{m,n,p,q,r,s,t} \frac{\partial^7 Q_{23}^i}{\partial X_m \partial X_n \partial X_p \partial X_q \partial X_r \partial X_s \partial X_t} H_m H_n H_p H_q H_r H_s H_t = 0$$

Using (4),

$$Q_{3,10}^i = \frac{1}{8!} \sum_{m,n,p,q,r,s,t,u} \frac{\partial^8 Q_{23}^i}{\partial X_m \partial X_n \partial X_p \partial X_q \partial X_r \partial X_s \partial X_t \partial X_u} H_m H_n H_p H_q H_r H_s H_t H_u = 0$$

And using (4) and (5),

$$Q_{3,11}^i = \frac{1}{9!} \sum_{m,n,p,q,r,s,t,u,v} \frac{\partial^9 Q_{23}^i}{\partial X_m \partial X_n \partial X_p \partial X_q \partial X_r \partial X_s \partial X_t \partial X_u \partial X_v} H_m H_n H_p H_q H_r H_s H_t H_u H_v = 0$$

We have then obtained

$$\begin{aligned} P_3^i &= Q_{31}^i + Q_{32}^i \\ &= 6L_i \textstyle\sum_{j,k=1}^d a_{ij} a_{ik} L_j^3 L_k^3 + \sum_{j,k,l=1}^d 6 a_{ij} a_{ik} a_{il} L_j^3 L_k^3 L_l^3. \end{aligned}$$

In particular, $\deg P_3^i \leq 9$. The expression obtained for P_3^i gives

$$P_4^i = \sum_{j=1}^{10} Q_{4j}^i \text{ with } \deg Q_{4j}^i = 2j + 7.$$

In order to determine P_4^i we need to compute the derivatives of Q_{31}^i and Q_{32}^i. We compute first the derivatives of Q_{31}^i.

$$\frac{\partial Q_{31}^i}{\partial X_l} = \sum_{j,k=1}^d (6 a_{ij} a_{ik} a_{il} L_j^3 L_k^3 + 18 a_{ij} a_{ik} a_{jl} L_i L_j^2 L_k^3 + 18 a_{ij} a_{ik} a_{kl} L_i L_j^3 L_k^2)$$

$$\begin{aligned} \frac{\partial^2 Q_{31}^i}{\partial X_l \partial X_m} = & \textstyle\sum_{j,k=1}^d (18 a_{ij} a_{ik} a_{il} a_{jm} L_j^2 L_k^3 + 18 a_{ij} a_{ik} a_{il} a_{km} L_j^3 L_k^2 + 18 a_{ij} a_{ik} a_{jl} a_{im} L_j^2 L_k^3 \\ & + 36 a_{ij} a_{ik} a_{jl} a_{jm} L_i L_j L_k^3 + 54 a_{ij} a_{ik} a_{jl} a_{km} L_i L_j^2 L_k^2 + 18 a_{ij} a_{ik} a_{kl} a_{im} L_j^3 L_k^2 \\ & + 54 a_{ij} a_{ik} a_{kl} a_{jm} L_i L_j^2 L_k^2 + 36 a_{ij} a_{ik} a_{kl} a_{km} L_i L_j^3 L_k). \end{aligned}$$

The derivative $\partial^3 Q_{31}^i / \partial X_l \partial X_m \partial X_n$ is a sum of terms of one of the following forms

$$\begin{array}{ll} 36 \sum_{j,k=1}^d a_{ij} a_{ik} a_{il} a_{jm} a_{jn} L_j L_k^3, & 54 \sum_{j,k=1}^d a_{ij} a_{ik} a_{il} a_{jm} a_{kn} L_j^2 L_k^2 \\ 36 \sum_{j,k=1}^d a_{ij} a_{ik} a_{jl} a_{jm} a_{jn} L_i L_k^3, & 108 \sum_{j,k=1}^d a_{ij} a_{ik} a_{jl} a_{jm} a_{kn} L_i L_j L_k^2, \end{array}$$

up to a swift of j with k and a permutation of the set $\{l, m, n\}$. The derivative $\partial^4 Q_{31}^i / \partial X_l \partial X_m \partial X_n \partial X_p$ is a sum of terms of one of the following forms

$$36 \sum_{j,k=1}^d a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}L_k^3, \qquad 108 \sum_{j,k=1}^d a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{kp}L_jL_k^2$$
$$108 \sum_{j,k=1}^d a_{ij}a_{ik}a_{jl}a_{jm}a_{jn}a_{kp}L_iL_k^2, \quad 216 \sum_{j,k=1}^d a_{ij}a_{ik}a_{jl}a_{jm}a_{kn}a_{kp}L_iL_jL_k,$$

up to a swift of j with k and a permutation of the set $\{l, m, n, p\}$. The derivative $\partial^5 Q_{31}^i/\partial X_l \partial X_m \partial X_n \partial X_p \partial X_q$ is a sum of terms of one of the following forms

$$108 \sum_{j,k=1}^d a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}L_k^2, \qquad 216 \sum_{j,k=1}^d a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{kp}a_{kq}L_jL_k$$
$$216 \sum_{j,k=1}^d a_{ij}a_{ik}a_{jl}a_{jm}a_{jn}a_{kp}a_{kq}L_iL_k,$$

up to a swift of j with k and a permutation of the set $\{l, m, n, p, q\}$. The derivative $\partial^6 Q_{31}^i/\partial X_l \partial X_m \partial X_n \partial X_p \partial X_q \partial X_r$ is a sum of terms of one of the following forms

$$216 \sum_{j,k=1}^d a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}a_{kr}L_k, \quad 216 \sum_{j,k=1}^d a_{ij}a_{ik}a_{jl}a_{jm}a_{jn}a_{kp}a_{kq}a_{kr}L_i,$$

up to a swift of j with k and a permutation of the set $\{l, m, n, p, q, r\}$. The derivative $\partial^7 Q_{31}^i/\partial X_l \partial X_m \partial X_n \partial X_p \partial X_q \partial X_r \partial X_s$ is a sum of terms of the following form

$$216 \sum_{j,k=1}^d a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}a_{kr}a_{ks},$$

up to a swift of j with k and a permutation of the set $\{l, m, n, p, q, r, s\}$.

We compute now the derivatives of Q_{32}^i.

$$\frac{\partial Q_{32}^i}{\partial X_m} = 18 \sum_{j,k,l} (a_{ij}a_{ik}a_{il}a_{jm}L_j^2L_k^3L_l^3 + a_{ij}a_{ik}a_{il}a_{km}L_j^3L_k^2L_l^3 + a_{ij}a_{ik}a_{il}a_{lm}L_j^3L_k^3L_l^2)$$

The derivative $\partial^2 Q_{32}^i/\partial X_m \partial X_n$ is a sum of terms of one of the following forms

$$36 \sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}L_jL_k^3L_l^3, \quad 54 \sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{kn}L_j^2L_k^2L_l^3$$

up to a permutation of the set $\{j, k, l\}$ and a permutation of the set $\{m, n\}$. The derivative $\partial^3 Q_{32}^i/\partial X_m \partial X_n \partial X_p$ is a sum of terms of one of the following forms

$$36 \sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}L_k^3L_l^3, \qquad 108 \sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{kp}L_jL_k^2L_l^3$$
$$162 \sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{kn}a_{lp}L_j^2L_k^2L_l^2,$$

up to a permutation of the set $\{j, k, l\}$ and a permutation of the set $\{m, n, p\}$. The derivative $\partial^4 Q_{32}^i/\partial X_m \partial X_n \partial X_p \partial X_q$ is a sum of terms of one of the following forms

$$108\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}L_k^2L_l^3, \qquad 216\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{kp}a_{kq}L_jL_kL_l^3$$
$$324\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{kp}a_{lq}L_jL_k^2L_l^2,$$

up to a permutation of the set $\{j,k,l\}$ and a permutation of the set $\{m,n,p,q\}$. The derivative $\partial^5 Q_{32}^i/\partial X_m\partial X_n\partial X_p\partial X_q\partial X_r$ is a sum of terms of one of the following forms

$$216\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}a_{kr}L_kL_l^3, \qquad 324\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}a_{lq}L_k^2L_l^2$$
$$648\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{kp}a_{kq}a_{lr}L_jL_kL_l^2,$$

up to a permutation of the set $\{j,k,l\}$ and a permutation of the set $\{m,n,p,q,r\}$. The derivative $\partial^6 Q_{32}^i/\partial X_m\partial X_n\partial X_p\partial X_q\partial X_r\partial X_s$ is a sum of terms of one of the following forms

$$216\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}a_{kr}a_{ks}L_l^3, \qquad 648\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}a_{lq}a_{ks}L_kL_l^2$$
$$1296\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{kp}a_{kq}a_{lr}a_{ls}L_jL_kL_l,$$

up to a permutation of the set $\{j,k,l\}$ and a permutation of the set $\{m,n,p,q,r,s\}$. The derivative $\partial^7 Q_{32}^i/\partial X_m\partial X_n\partial X_p\partial X_q\partial X_r\partial X_s\partial X_t$ is a sum of terms of one of the following forms

$$648\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}a_{kr}a_{ks}a_{lt}L_l^2, \quad 1296\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}a_{lq}a_{ks}a_{lt}L_kL_l$$

up to a permutation of the set $\{j,k,l\}$ and a permutation of the set $\{m,n,p,q,r,s,t\}$. The derivative $\partial^8 Q_{32}^i/\partial X_m\partial X_n\partial X_p\partial X_q\partial X_r\partial X_s\partial X_t\partial X_u$ is a sum of terms of the form

$$1296\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}a_{kr}a_{ks}a_{lt}a_{lu}L_l,$$

up to a permutation of the set $\{j,k,l\}$ and a permutation of the set $\{m,n,p,q,r,s,t,u\}$. The derivative $\partial^9 Q_{32}^i/\partial X_m\partial X_n\partial X_p\partial X_q\partial X_r\partial X_s\partial X_t\partial X_u\partial X_v$ is a sum of terms of the form

$$1296\sum_{j,k,l} a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}a_{kr}a_{ks}a_{lt}a_{lu}a_{lv},$$

up to a permutation of the set $\{j,k,l\}$ and a permutation of the set $\{m,n,p,q,r,s,t,u,v\}$.

We compute now the homogeneous summands of P_4^i.

Using (3),

$$Q_{41}^i = \sum_{l=1}^{d} \frac{\partial Q_{31}^i}{\partial X_l} H_l = \sum_{j,k,l=1}^{d} 6a_{ij}a_{ik}a_{il}L_j^3L_k^3L_l^3.$$

Using (3), (4) and (5), we obtain

$$Q_{4j}^i = 0, \quad \forall j = 2, \dots, 10.$$

We have then

$$P_4^i = Q_{41}^i = \sum_{j,k,l=1}^{d} 6 a_{ij} a_{ik} a_{il} L_j^3 L_k^3 L_l^3,$$

hence P_4^i is a homogeneous polynomial of degree 9. The expression obtained for P_4^i gives

$$P_5^i = \sum_{j=1}^{9} Q_{5j}^i \text{ with } \deg Q_{5j}^i = 2j + 9.$$

In order to determine P_5^i we compute the derivatives of Q_{41}^i.

$$\frac{\partial Q_{41}^i}{\partial X_m} = 18 \sum_{j,k,l=1}^{d} (a_{ij} a_{ik} a_{il} a_{jm} L_j^2 L_k^3 L_l^3 + a_{ij} a_{ik} a_{il} a_{km} L_j^3 L_k^2 L_l^3 + a_{ij} a_{ik} a_{il} a_{lm} L_j^3 L_k^3 L_l^2).$$

The derivative $\partial^2 Q_{41}^i / \partial X_m \partial X_n$ is a sum of terms of one of the following forms

$$36 \sum_{j,k,l=1}^{d} a_{ij} a_{ik} a_{il} a_{jm} a_{jn} L_j L_k^3 L_l^3, \quad 54 \sum_{j,k,l=1}^{d} a_{ij} a_{ik} a_{il} a_{jm} a_{kn} L_j^2 L_k^2 L_l^3,$$

up to a permutation of the set $\{j, k, l\}$ and a permutation of the set $\{m, n\}$. The derivative $\partial^3 Q_{41}^i / \partial X_m \partial X_n \partial X_p$ is a sum of terms of one of the following forms

$$\begin{array}{ll} 36 \sum_{j,k,l=1}^{d} a_{ij} a_{ik} a_{il} a_{jm} a_{jn} a_{jp} L_k^3 L_l^3, & 108 \sum_{j,k,l=1}^{d} a_{ij} a_{ik} a_{il} a_{jm} a_{jn} a_{kp} L_j L_k^2 L_l^3, \\ 162 \sum_{j,k,l=1}^{d} a_{ij} a_{ik} a_{il} a_{jm} a_{kn} a_{lp} L_j^2 L_k^2 L_l^2, & \end{array}$$

up to a permutation of the set $\{j, k, l\}$ and a permutation of the set $\{m, n, p\}$. The derivative $\partial^4 Q_{41}^i / \partial X_m \partial X_n \partial X_p \partial X_q$ is a sum of terms of one of the following forms

$$\begin{array}{ll} 108 \sum_{j,k,l=1}^{d} a_{ij} a_{ik} a_{il} a_{jm} a_{jn} a_{jp} a_{kq} L_k^2 L_l^3, & 216 \sum_{j,k,l=1}^{d} a_{ij} a_{ik} a_{il} a_{jm} a_{jn} a_{kp} a_{kq} L_j L_k L_l^3, \\ 324 \sum_{j,k,l=1}^{d} a_{ij} a_{ik} a_{il} a_{jm} a_{kn} a_{lp} a_{jq} L_j L_k^2 L_l^2, & \end{array}$$

up to a permutation of the set $\{j, k, l\}$ and a permutation of the set $\{m, n, p, q\}$. The derivative $\partial^5 Q_{41}^i / \partial X_m \partial X_n \partial X_p \partial X_q \partial X_r$ is a sum of terms of one of the following forms

$$\begin{array}{ll} 216 \sum_{j,k,l=1}^{d} a_{ij} a_{ik} a_{il} a_{jm} a_{jn} a_{jp} a_{kq} a_{kr} L_k L_l^3, & 324 \sum_{j,k,l=1}^{d} a_{ij} a_{ik} a_{il} a_{jm} a_{jn} a_{jp} a_{kq} a_{lr} L_k^2 L_l^2, \\ 648 \sum_{j,k,l=1}^{d} a_{ij} a_{ik} a_{il} a_{jm} a_{jn} a_{kp} a_{kq} a_{lr} L_j L_k L_l^2, & \end{array}$$

up to a permutation of the set $\{j,k,l\}$ and a permutation of the set $\{m,n,p,q,r\}$. The derivative $\partial^6 Q^i_{41}/\partial X_m \partial X_n \partial X_p \partial X_q \partial X_r \partial X_s$ is a sum of terms of one of the following forms

$$648 \sum_{j,k,l=1}^d a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}a_{kr}a_{ls}L_k L_l^2, \qquad 216 \sum_{j,k,l=1}^d a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}a_{kr}a_{ks}L_l^3,$$
$$1296 \sum_{j,k,l=1}^d a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{kp}a_{kq}a_{lr}a_{ls}L_j L_k L_l,$$

up to a permutation of the set $\{j,k,l\}$ and a permutation of the set $\{m,n,p,q,r,s\}$. The derivative $\partial^7 Q^i_{41}/\partial X_m \partial X_n \partial X_p \partial X_q \partial X_r \partial X_s \partial X_t$ is a sum of terms of one of the following forms

$$648 \sum_{j,k,l=1}^d a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}a_{kr}a_{ls}a_{kt}L_l^2, \; 1296 \sum_{j,k,l=1}^d a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}a_{kr}a_{ls}a_{lt}L_k L_l,$$

up to a permutation of the set $\{j,k,l\}$ and a permutation of the set $\{m,n,p,q,r,s,t\}$. The derivative $\partial^8 Q^i_{41}/\partial X_m \partial X_n \partial X_p \partial X_q \partial X_r \partial X_s \partial X_t \partial X_u$ is a sum of terms of the following form

$$1296 \sum_{j,k,l=1}^d a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}a_{kr}a_{ls}a_{kt}a_{lu}L_l,$$

up to a permutation of the set $\{j,k,l\}$ and a permutation of the set $\{m,n,p,q,r,s,t,u\}$. The derivative $\partial^9 Q^i_{41}/\partial X_m \partial X_n \partial X_p \partial X_q \partial X_r \partial X_s \partial X_t \partial X_u \partial X_v$ is a sum of terms of the following form

$$1296 \sum_{j,k,l=1}^d a_{ij}a_{ik}a_{il}a_{jm}a_{jn}a_{jp}a_{kq}a_{kr}a_{ls}a_{kt}a_{lu}a_{lv},$$

up to a permutation of the set $\{j,k,l\}$ and a permutation of the set $\{m,n,p,q,r,s,t,u,v\}$. We have

$$Q^i_{5j} = \frac{1}{j!} \sum_{n_1,\ldots,n_j=1}^d \frac{\partial^j Q^i_{41}}{\partial X_{n_1} \ldots \partial X_{n_j}} H_{n_1} \ldots H_{n_j}$$

and, using again (3), (4), (5), we obtain

$$Q^i_{5j} = 0, \text{ for all } \; j,$$

hence $P^i_5 = 0$, for all $i = 1 \ldots, d$, as wanted. $\square$

The following example shows that the result in Proposition 4 is optimal, in the sense that, under the hypothesis $(JH)^3 = 0$, we may not expect that the first zero term in the sequence (P^i_j) comes before the fifth for all $i = 1, \ldots, d$.

Example 5. We consider the polynomial map F as in Theorem 3 with $d = 5$ and

$$\begin{cases} L_1(X_1, X_2, X_3, X_4, X_5) &= a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5 \\ L_2(X_1, X_2, X_3, X_4, X_5) &= b_3X_3 + b_4X_4 + b_5X_5 \\ L_3(X_1, X_2, X_3, X_4, X_5 &= 0 \\ L_4(X_1, X_2, X_3, X_4, X_5) &= 0 \\ L_5(X_1, X_2, X_3, X_4, X_5) &= 0 \end{cases}$$

with $a_2, a_3, a_4, a_5, b_3, b_4, b_5$ complex parameters. We may check that the Jacobian matrix JH is nilpotent of degree 3. By computing the sequences of polynomials P_1^i, we obtain

$$\begin{aligned} P_2^1 &= 3a_2L_1^2L_2^3 + 3a_2^2L_1L_2^6 + a_2^3L_2^9 \\ P_3^1 &= 6a_2^2L_1L_2^6 + 6a_2^3L_2^9 \\ P_4^1 &= 6a_2^3L_2^9 \\ P_5^1 &= 0 \end{aligned}$$

Clearly, $P_2^2 = 0$ and $P_1^3 = P_1^4 = P_1^5 = 0$. Then, choosing such an F with $a_2 \neq 0$ and L_2 not identically zero, we obtain an example of a polynomial map F such that the Jacobian matrix JH is nilpotent of degree 3 and at least one P_4^i is not zero. Hence the inverse map F^{-1} has degree equal to 9. The map $G = F^{-1}$ is given by

$$\begin{cases} G_1(Y) &= Y_1 - L_1(Y)^3 + 3a_2L_1(Y)^2L_2(Y)^3 - 3a_2^2L_1(Y)L_2(Y)^6 + a_2^3L_2(Y)^9 \\ G_2(Y) &= Y_2 - L_2(Y)^3 \\ G_3(Y) &= Y_3 \\ G_4(Y) &= Y_4 \\ G_5(Y) &= Y_5 \end{cases}$$

where $Y := (Y_1, Y_2, Y_3, Y_4, Y_5)$.

The result obtained in Theorem 3 leads us to formulate the following conjecture.

Conjecture 6. *Let $F : K^d \to K^d$ be a polynomial map of the form*

$$\begin{cases} F_1(X_1, \ldots, X_d) &= X_1 + H_1(X_1, \ldots, X_d) \\ F_2(X_1, \ldots, X_d) &= X_2 + H_2(X_1, \ldots, X_d) \\ &\vdots \\ F_d(X_1, \ldots, X_d) &= X_d + H_d(X_1, \ldots, X_d), \end{cases}$$

where $H_i(X_1, \ldots, X_d) = L_i(X_1, \ldots, X_d)^3$ and $L_i(X_1, \ldots, X_d) = a_{i1}X_1 + \cdots + a_{id}X_d$, $1 \leq i \leq d$. We consider the Jacobian matrix JH of $H := (H_1, \ldots, H_d)$. For each $i = 1, \ldots, d$, we consider the polynomial sequence (P_j^i) defined in the following way

$$\begin{aligned} P_0^i(X_1, \ldots, X_d) &= X_i, \\ P_1^i(X_1, \ldots, X_d) &= H_i, \end{aligned}$$

and, assuming P^i_{j-1} is defined,

$$P^i_j(X_1,\dots,X_d) = P^i_{j-1}(F_1,\dots,F_d) - P^i_{j-1}(X_1,\dots,X_d).$$

Let g be an integer, $1 \le g \le d$. If $(JH)^g = 0$, then $P^i_{(3^{g-1}+1)/2} = 0$, for all $i = 1,\dots,d$, F is invertible and the inverse of F has maximal degree at most equal to 3^{g-1}.

Remark 7. By Theorem 2, the condition $P^i_{(3^{g-1}+1)/2} = 0$, for all $i = 1,\dots,d$, implies that F is invertible. Since, for F as in Conjecture 6, $\det(JF) = 1$ implies $(JH)^d = 0$, and taking into account Proposition 1, Conjecture 6 implies the Jacobian conjecture for complex polynomial maps.

Remark 8. Conjecture 6 is true for

1) $g = 1$: since H_i is homogeneous, $JH = 0$ implies $H = 0$, hence $P^i_1 = 0$, $\forall i = 1,\dots,d$ and $F = Id$.

2) $g = 2$: if $(JH)^2 = 0$, we have $\sum_{j=1}^d \frac{\partial H_i}{\partial X_j}\frac{\partial H_j}{\partial X_k} = 0$, $\forall\, i,k = 1,\dots,d$. Multiplying by X_k and summing up from $k = 1$ to d, we obtain $\sum_{j=1}^d \frac{\partial H_i}{\partial X_j} H_j = 0$, $\forall\, i$. Then F is a quasi-translation, $P^i_2 = 0$, for all $i = 1,\dots,d$, and $F^{-1} = X - H$ (see [1] Proposition 9).

3) $g = 3$: this is Proposition 4.

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